Dynamical percolation model of oil displacement by water in the oil reservoir

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Abstract. A quasi-quantum-mechanical percolation model, which imitates the process of oil displacement by water from the oil reservoir, is suggested. The Hamiltonian describing the irreversible evolution process of the percolation lattice, which models the oil reservoir, is constructed. With the help of the computer calculations for the rectangular plane lattice, the time development of the process under consideration is observed. We consider the critical behaviour within the time interval $t_1 \leqslant t \leqslant t_2$ where time moment t_1 corresponds to the percolation phase transition (when water appears in the production wells for the first time) and time moment t_2 corresponds to the moment when the development of the percolation process is stopped (when only water begins to flow through the production wells).

1. Introduction

In the past several years the mathematical apparatus of percolation theory has been developed for the description of the results of the oil reservoir irrigation process [1-4]. The development of such a new approach has very significant applications as it allows us to consider the oil production problem from the general position of the second-kind phase transition theory. The previous investigations [1-4] concerned only the static aspects as they were devoted to the description of the oil reservoir irrigation process results. The approach proposed in this paper describes the time development and therefore gives the dynamical picture of the irrigation process. This approach can be called quasi-quantum-mechanical because the information about the oil reservoir at the arbitrary time t is given by some state function $\Psi(t)$ which has a great deal in common with the quantum mechanical wavefunction. The evolution of $\Psi(t)$ is determined by the equation of the Schrödinger type where the operator playing the role of the Hamiltonian is of the special choice. It is worth noting that some dynamical theories of the growth processes based on the Marcovian field-theoretical approach have already been developed in [5, 6], but are not acceptable for our purposes.

2. Dynamical model of the percolation process

Let us consider the oil and water as immiscible and incompressible fluids and the oil reservoir as a network of pores containing oil which are connected by narrow capillary channels. This system is modelled by the integer lattice, the sites of which are identified

with pores and the bonds with capillary channels of the oil reservoir. The probability field is introduced on the lattice by ascribing to every lattice site with number j the probability P_j that the water having reached this site will be able to displace the oil from it. The probability P_j directly depends on the local geological and physical-chemical properties of the oil reservoir and on the pressure gradient. The important characteristic of such a random field is the potential percolation probability P which is equal to the number of the potentially available sites (the sites from which it is possible to displace the oil by the water, or for which $P_j \geq P_c$) divided by the total number of the lattice sites. The percolation lattice constructed in such a way is the typical structure in which the particular second-kind phase transition, connected with the phenomenon of percolation through a disordered conducting media, can occur.

In the concrete calculations the rectangular plane lattices of the final size are usually considered. The upper row of such a lattice is identified with the injection wells and the lower row with the production wells. In such a system we interpret the fact that the water reaches the production wells as the realisation of the percolation. It is also possible to use another variant of the injection and production wells arrangement.

The rather complicated mathematical apparatus for the description of the dynamical percolation will be offered below. The necessity of its usage is connected with the need to obtain not only the numerical results but also the qualitative analytical ones. The usage of the spin algebra formalism for this purpose will allow us to exploit the properly developed methods of the quantum theory of magnetism in order to obtain the analytical results afterwards. Thus the state of every site j of the percolation lattice will be described by the double-meaning spinor function $\chi_j^{1,2}$ where $\chi_j^1 = (1, 0)_j$ corresponds to the site occupied by the oil and $\chi_j^2 = (0, 1)_j$ to the site occupied by the water. The state of the whole oil reservoir at the time moment t is given by the function:

$$\Psi(t) = \prod_{j=1}^{N} \chi_j^{m_j}(t) \tag{1}$$

where N is the total number of lattice sites and $m_j = 1, 2$. The total space $\mathcal{H}$ of the functions $\chi_j^{m_j}$ of N sites consists of 2^N functions of the kind (1), each of them differing from the others by its dependence on its own set $\{m_j, j = 1, \ldots, N\}$. Since the spinor functions form the orthonormal basis $\langle \chi_j^m | \chi_j^n \rangle = \delta_{mn}$, where $\langle \, | \, \rangle$ is the scalar multiplication symbol, the functions $\Psi(t)$ possess the same property. In the space of the spinor functions $\chi_j^{1,2}$ it is possible to introduce the basis spin operators S_j^x, S_j^y, S_j^z of the spin $S = \frac{1}{2}$ or their following linear combinations:

$$
\begin{aligned}
\mu_j^{11} &= \tfrac{1}{2} + S_j^z & \mu_j^{22} &= \tfrac{1}{2} - S_j^z \\
\mu_j^{21} &= S_j^x - iS_j^y & \mu_j^{12} &= S_j^x + iS_j^y
\end{aligned}
\tag{2}
$$

which are more convenient for our purposes. The operators μ_j^{ab} $(a, b = 1, 2)$ possess the following properties:

$$\mu_j^{ab}\mu_j^{cd} = \delta_{bc}\mu_j^{ad} \qquad \mu_j^{ab}\chi_j^c = \delta_{bc}\chi_j^a \tag{3}$$

Thus it can be seen here that, for example, the operator μ_j^{21} transfers the state χ_j^1 into the state χ_j^2 (oil into water) and μ_j^{22} into zero, while the operator μ_j^{22} does not change the state χ_j^2 and transfers the state χ_j^1 into zero and so on.

Let us define the operators:

$$\pi_j = \mu_j^{22} + \mu_j^{21} \qquad \Omega_j = \sum_k \mu_{j+k}^{22} \tag{4}$$

where the sum over k is the sum over the site j nearest neighbours $(k = j \pm 1)$. These operators have the following properties:

$$(\mu_j^{22})^2 = \mu_j^{22} \qquad (\pi_j)^2 = \pi_j \qquad \mu_j^{22}\pi_j = \pi_j \qquad \pi_j\mu_j^{22} = \mu_j^{22} \qquad (5)$$

Thus the operator $H_j = \pi_j\Omega_j$ transfers the state χ_j^1 into χ_j^2 (oil into water) if in the nearest environment of the site j there is a site $j + k$ which is in the state χ_{j+k}^2 (i.e. is occupied by the water). Now it is possible to come to the conclusion that the operator

$$H = \sum_{\langle j \rangle} H_j \qquad (6)$$

will define the evolution process of the percolation lattice (oil reservoir) if we consider it as the Hamiltonian operator of the Schrödinger equation

$$\frac{d\Psi(t)}{dt} = \frac{1}{\tau} H\Psi(t) \qquad (7)$$

where $\langle j \rangle$ means the totality of the potentially opened sites from which the oil can be displaced under the given geological and physical-chemical conditions, and τ is the average time between two elementary acts of oil displacements from the neighbouring pores. The Hamiltonian in form (4) and (6) is not the Hermitian operator and this fact ensures the irreversibility of the evolution process which is described by equation (7) (irreversibility of the oil reservoir irrigation).

The solution of equation (7) can be presented in the form

$$\Psi(t) = S(t, 0)\Psi(0) \qquad S(t, 0) = \exp\left(\frac{tH}{\tau}\right) \qquad (8)$$

$$S(t, 0) = \sum_{n=0} S_n(t, 0) \qquad S_n(t, 0) = \frac{1}{n!}\left(\frac{tH}{\tau}\right)^n \qquad (9)$$

where $\Psi(0)$ is defined by the initial state of the percolation lattice. Expansion (9) of exponent (8) can be formally extended till infinity. However, as will be seen later, Hamiltonian (6) of the model under consideration is constructed in such a way that the percolation lattice evolution is stopped at some stage. From the mathematical point of view it is equivalent to the condition

$$S_{n_c+1}(t, 0)\Psi(0) = S_{n_c}(t, 0)\Psi(0) \qquad (10)$$

for some critical value of n_c.

If the potential percolation probability P is smaller than some critical probability P_c which is typical for the percolation lattice of the chosen kind, then the process of percolation in such a system does not occur with statistical validity (the values of P_c can be found in [5] for different lattices). In this case equation (7) describes the irrigation process only for that part of the percolation lattice which lies near to the first row of sites (injection wells). When percolation takes place $(P \geqslant P_c)$ there are two notable events. The first event is the reaching by the water at time t_1 of the lower lattice border sites corresponding to the row of production wells. The second event is the end of the oil displacement process at time t_2. It is evident that $t_2 \geqslant t_1$.

Now it is necessary to define the rules of the percolation lattice state functions normalisation when the operators $S(t, 0)$ have already acted. In order to explain the difficulties and the suggested solution of the problem it is convenient to consider a fragment of the plane percolation lattice as in figure 1, where the sites potentially
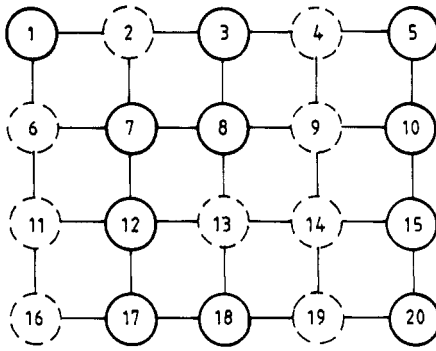
Figure 1. A fragment of the percolation lattice. The sites potentially available for percola-
tion are encircled with bold lines while unavailable sites are encircled with broken lines.

available for percolation are encircled by bold lines. Their number corresponds to the
potential percolation probability $P = 0.55$. For the configuration of percolation sites
shown in figure 1 the Hamiltonian is defined as

$$H = \sum_{\langle j \rangle} H_j = H_1 + H_3 + H_5 + H_7 + H_8 + H_{10} + H_{12} + H_{15} + H_{17} + H_{18} + H_{20}. \tag{11}$$

There are three sites in the upper row which are irrigated at the initial moment so that
the initial state of the oil reservoir is given by the function

$$\Psi(0) = \chi_1^2 \chi_3^2 \chi_5^2 \prod_j{}^* \chi_j^1. \tag{12}$$

In our notation the asterisk over the product sign means that in the product $\prod_j^* \chi^1$
there are the state functions χ^1 for all sites of the percolation lattice with the exception
of sites whose state functions are written on the left side of the product sign $\prod_j^*$.

Let us consider the action of Hamiltonian (11) on the oil reservoir state function
(12). Guided by definitions (3), (4) and (6), we obtain:

$$\psi(\tau) = H\psi(0) = \chi_1^2 \chi_3^2 \chi_1^2 (\chi_8^2 \chi_{10}^2 + \chi_{10}^2 \chi_8^2) \prod_j{}^* \chi_j^1 \tag{13}$$

$$\psi(2\tau) = \tfrac{1}{2} H^2 \Psi(0) = \tfrac{1}{2} \chi_1^2 \chi_3^2 \chi_5^2 (\chi_7^2 \chi_8^2 \chi_{10}^2 \chi_{15}^1 + \chi_{10}^2 \chi_{15}^2 \chi_8^2 \chi_7^1) \prod_j{}^* \chi_j^1. \tag{14}$$

It is obvious that, using the standard quantum-mechanical normalisation of the state
functions $\Psi(\tau)$, $\Psi(2\tau)$ and so on, then it can be found that, for example, at time $t = 2\tau$
there will be twice as much water in states χ_1^2, χ_3^2 and χ_5^2 than in states χ_7^2, χ_8^2, χ_{10}^2
and χ_{15}^2. Hence it follows that, the more ramified is the network, the greater are the
differences in the amount of water between the common channel (in the case under
consideration for $j = 1, 3, 5$) and the ramified part (for $j = 7, 8, 10, 15$) of the percolation
lattice. This is the reason why the standard quantum-mechanical normalisation of the
oil reservoir state functions is not suitable when one has accepted the interpretation
of the states χ_j^2 and χ_j^2 as the symbols which describe the only situation when the site
j is completely occupied by the immiscible and incompressible fluids—oil or water.

In accordance with the accepted model of the oil reservoir description, state
functions (13) and (14) must describe such oil reservoir states which have the same
amount of water during the irrigation process in the sites with $j = 1, 3, 5, 7$ at the time
$t = \tau$ and in the sites with $j = 1, 3, 5, 7, 8, 10, 15$ at the time $t = 2\tau$ when the other sites

are still occupied by the oil. The following state functions, taken in the form (1), completely correspond to this situation:

$$\Psi(\tau) = \chi_1^2 \chi_3^2 \chi_5^2 \chi_8^2 \chi_{10}^2 \prod_j{}^* \chi_j^1 \tag{13'}$$

$$\Psi(2\tau) = \chi_1^2 \chi_3^2 \chi_5^2 \chi_8^2 \chi_{10}^2 \chi_{15}^2 \prod_j{}^* \chi_j^1. \tag{14'}$$

It is possible to reach the identification of expressions (13) and (13'), (14) and (14'), and so on, if we consider that the sum of two arbitrary state functions (taken at the same time moment) $\Psi_1 = \Pi_{j=1}^N \chi_j^{m_j}$ and $\Psi_2 = \Pi_{j=1}^N \chi_j^{n_j}$ is the state function $\Psi = \Pi_{j=1}^N \chi_j^{p_j}$ where for every j the function $\chi_j^{p_j} = \chi_j^{m_j} \oplus \chi_j^{n_j}$ is defined from $\chi_j^{m_j}$ and $\chi_j^{n_j}$ with the help of the following binary composition rules:

$$\chi_j^1 \oplus \chi_j^1 = \chi_j^1 \qquad \chi_j^2 \oplus \chi_j^2 = \chi_j^2 \qquad \chi_j^1 \oplus \chi_j^2 = \chi_j^2. \tag{15}$$

When such composition rules are taken into account the set of 2^N possible state functions of the kind (1) forms at every time moment the commutative semigroup with the identity element $\Psi_1 = \Pi_{j=1}^N \chi_j^1$. (If we introduce in this semigroup the multiplication operation through the summation operation, as is usually done in set theory, then the functions set of the kind (1) would form the Boolean algebra.) The attaching to this set (1) the properties of the commutative semigroup ensures a normalisation of such state functions to be adequate for the oil reservoir model under consideration. This normalisation will be designated below by the letter S as composition rules (15) are the summation in the set theory.

Solution (9) of equation (8) displays that the percolation lattice evolution is defined by the action of the Hamiltonian H to the different powers on the percolation lattice state function. In order to analyse that action it is necessary to determine the main properties of the above-defined normalisation. Before doing that, we introduce several definitions.

Definition D1. Let the set $\{j\}$ of L sites be picked out by some way. On this set we define the particular and homogeneous p-forms of the operators μ_j^{22}:

$$t_p(\mu_{j_1}^{22}, \ldots, \mu_{j_p}^{22}) = \varepsilon_{j_1 \ldots j_p}^{j_1 \ldots j_p} \prod_{l=1}^p \mu_{j_l}^{22} \tag{16}$$

$$t_p(\mu^{22}) = \sum_{\{j\}} t_p(\mu_{j_1}^{22}, \ldots, \mu_{j_p}^{22}) \tag{17}$$

where $\varepsilon_{j_1 \ldots j_p}^{j_1 \ldots j_p}$ is the generalised Kronecker delta of rank $2p$ ($p \leq L$). Due to the multiplication property (5) of operators μ_j^{22} any function $f(\mu^{22})$ defined with their help can be presented as an expansion in $t_p(\mu_{j_1}^{22}, \ldots, \mu_{j_p}^{22})$:

$$f(\mu^{22}) = \sum_{p=1}^L \sum_{j_1} \ldots \sum_{j_p} C(j_1, \ldots, j_p) t_p(\mu_{j_1}^{22}, \ldots, \mu_{j_p}^{22}). \tag{18}$$

As will be shown below, it is possible to suppose that under the S-normalisation conditions the coefficients $C(j_1, \ldots, j_p)$ can be equal only to 0 or 1.

Definition D2. Let us define the effective Hamiltonian of order p by the expression

$$H_p = \sum_{\langle j_1, \ldots, j_p \rangle} \sum H_{\langle j_1, \ldots, j_p \rangle}^{(p)} \tag{19}$$

where

$$H^{(p)}_{\langle j_1, \ldots, j_p \rangle} = \Omega_{j_1 \ldots j_p} \prod_{l=1}^{P} \pi_{j_l}$$

$$\Omega_{j_1 \ldots j_p} = \sum_{\langle j_1 \ldots j_p \rangle^*} \mu^{22}_{\langle j_1} \mu^{22}_{j_2} \ldots \mu^{22}_{j_p)^*}$$

where $\langle j_1 \ldots j_p \rangle$ designates the cluster which consists of p neighbouring opened for percolation sites and $\langle j_1 \ldots j_p \rangle^*$ designates the totality of sites which are the nearest neighbours of cluster $\langle j_1 \ldots j_p \rangle$. In particular, in the terms of the given definition, the percolation lattice Hamiltonian (6) is $H = H_1$. Let us define the following auxiliary operator:

$$\theta_{j_1 \ldots j_p, j_1 \ldots j_q} = \Omega_{j_1 \ldots j_p} - \sum_{l=1}^{q} \mu^{22}_{j_l}. \tag{20}$$

Definition D3. Let us pick out the subspaces $\mathcal{H}_j$, $\mathcal{H}_{j_1 j_2}$, and so on, from the percolation lattice state functions total space $\mathcal{H}$. They possess the following properties: the subspace $\mathcal{H}_j$ consists of the functions which always have as factor the function χ_j^2 and have not the function χ_j^1. The subspaces $\mathcal{H}_{j_1 j_2}$ consist of the functions which always have as factor the production $\chi_{j_1}^2 \chi_{j_2}^2$ and so on. It is evident that $\mathcal{H}_j \in \mathcal{H}$, $\mathcal{H}_{j_1 j_2} \in \mathcal{H}_{j_1}$, $\mathcal{H}_{j_2}$. The subspaces $\mathcal{H}_{j_1 \ldots j_p}$ can be symbolically defined in the following way:

$$\mathcal{H}_{j_1 \ldots j_p} = \prod_{l=1}^{p} \mu^{22}_{j_l} \mathcal{H}. \tag{21}$$

Let us now formulate the S-normalisation properties whose proofs are given in the appendix.

Property P1. The matrix elements of the Hamiltonian H in the basis (1) are real numbers. The S-normalisation result does not depend on the numerical coefficients of the items.

Property P2. $S[t_p(\mu^{22}) \oplus t_p(\mu^{22}_{j_1}, \ldots, \mu^{22}_{j_p})] \Psi = t_p(\mu^{22}) \Psi.$

Property P3. $S[t_p(\mu^{22}) \oplus t_{p+q}(\mu^{22})] \Psi = t_p(\mu^{22}) \Psi \qquad (q \geq 0).$

Property P4. $SH_j^2 \Psi = H_j \Psi.$

Property P5. $SAB\Psi \in (A + B)\Psi$ where two cluster sets, on which the operators A and B are defined, have not the same sites.

Property P6. $SH^2 \Psi = \left[H + \sum_{\langle j_1 j_2 \rangle} H^{(2)}_{\langle j_1 j_2 \rangle} \right] \Psi.$

The essential feature of the proposed scheme is in the finite time evolution. Let us anticipate the proof of this fact by the statement about the general structure of the result of the operator H^n ($n \geq 1$) action on the functions $\Psi \in \mathcal{H}$ within the bounds of the S-normalisation.

Let us now prove two statements that are important for the dynamic percolation model theory.

Lemma. The general structure of the result of the operator H action on the state function Ψ within the S-normalisation is:

$$SH^n\Psi = \sum_{p=0}^{n} H_p\Psi. \tag{22}$$

In order to prove this lemma let us use the mathematical induction method. When $n = 1$ this lemma is satisfied identically and when $n = 2$ its proof is given by property P6. Assuming now that the lemma is true for some power n let us prove it for the power $n + 1$. By presenting the product $H^{n+1} = \frac{1}{2}(HH^n + H^nH)$ as the sum of three existing possibilities:

$$SH^{n+1}\Psi = S \sum_{p=0}^{n} R\Psi \tag{23}$$

where

$$R = \left(\sum_{\langle j_1 \ldots j \ldots j_p \rangle} \oplus \sum_{\substack{\langle j \rangle \notin \langle j_1 \ldots j_p \rangle \\ \langle j \rangle \notin \langle j_1 \ldots j_p \rangle^*}} \oplus \sum_{\langle j \rangle \in \langle j_1 \ldots j_p \rangle^*} \right) \tfrac{1}{2}[H^{(1)}_{\langle j \rangle}H^{(p)}_{\langle j_1 \ldots j_p \rangle} + H^{(p)}_{\langle j_1 \ldots j_p \rangle}H^{(1)}_{\langle j \rangle}]$$

where the site j belongs to the cluster $\langle j_1 \ldots j_p \rangle$; the site j does not belong to the clusters $\langle j_1 \ldots j_p \rangle$ and $\langle j_1 \ldots j_p \rangle^*$ and the site j belongs to the cluster $\langle j_1 \ldots j_p \rangle^*$.

Let us consider the first sum of expression (23), and let, for example, the site $j + n$, from the nearest environment of site j, belong to the cluster $\langle j_1 \ldots j_p \rangle$ (our discussions do not change when any number of sites from the nearest environment of the site j belongs to the cluster $\langle j_1 \ldots j_p \rangle$). Then we obtain:

$$\tfrac{1}{2}[H^{(1)}_{\langle j \rangle}H^{(p)}_{\langle j_1 \ldots j_p \rangle} + H^{(p)}_{\langle j_1 \ldots j_p \rangle}H^{(1)}_{\langle j \rangle}]$$
$$= H^{(p)}_{\langle j_1 \ldots j \ldots j+k \ldots j_p \rangle}(\tfrac{1}{2} + \theta_{j,j+k}) + \tfrac{1}{2}\pi_{j_1} \ldots \pi_j \ldots \mu^{22}_{j+k} \ldots \pi_{j_p} \Omega_{j_1 \ldots j \ldots j+k \ldots j_p}. \tag{24}$$

Taking into account properties P1 and P3, and also the fact that for every m the expression $\mu^{22}_j \chi^m_j \in \pi_j \chi^m_j$ is valid, it is possible to obtain that:

$$S\tfrac{1}{2}[H^{(1)}_{\langle j \rangle}H^{(p)}_{\langle j_1 \ldots j_p \rangle} \oplus H^{(p)}_{\langle j_1 \ldots j_p \rangle}H^{(1)}_{\langle j \rangle}]\Psi = H^{(p)}_{\langle j_1 \ldots j_p \rangle}\Psi. \tag{25}$$

That is why the first sum in expression (23) is equal to $H_p\Psi$.

Let us consider the second sum of expression (23). Because of the fact that there are not the same sites in the definition of the operators $H^{(1)}_{\langle j \rangle}$ and $H^{(p)}_{\langle j_1 \ldots j_p \rangle}$ it is possible to use property P5 from which it follows that:

$$S\tfrac{1}{2}[H^{(1)}_{\langle j \rangle}H^{(p)}_{\langle j_1 \ldots j_p \rangle} \oplus H^{(p)}_{\langle j_1 \ldots j_p \rangle}H^{(1)}_{\langle j \rangle}]\Psi \in [H^{(1)}_{\langle j \rangle} + H^{(p)}_{\langle j_1 \ldots j_p \rangle}]\Psi. \tag{26}$$

So it can be seen that the action of items, which are the members of the second sum in expression (23), has already been taken into account in the first and in the pth terms of series (23) and that is why they can be omitted in the S-normalisation.

Let us consider the third sum of expression (23). And let, for example, only one site $j + k$ from the nearest environment of the site j be the member of the cluster $\langle j_1 \ldots j_p \rangle$ (our discussions do not change when any number of the neighbours of site j are the members of the cluster $\langle j_1 \ldots j_p \rangle$). Thus we obtain:

$$\tfrac{1}{2}[H^{(1)}_{\langle j \rangle}H^{(p)}_{\langle j_1 \ldots j_p \rangle} + H^{(p)}_{\langle j_1 \ldots j_p \rangle}H^{(1)}_{\langle j \rangle}]$$
$$= \pi_{j_1} \ldots \pi_j \ldots \pi_{j+k} \ldots \pi_{j_p} \theta_{j_1 \ldots j+k \ldots j_p, j} \theta_{j,j+k}$$
$$+ \tfrac{1}{2}\pi_{j_1} \ldots \pi_j \ldots (\pi_{j+k} + \mu^{22}_{j+k}) \ldots \pi_{j_p} \theta_{j_1 \ldots j+k \ldots j_p, j}$$
$$+ \tfrac{1}{2}\pi_{j_1} \ldots (\pi_j + \mu^{22}_j) \ldots \pi_{j+k} \ldots \pi_{j_p} \theta_{j,j+k} + \tfrac{1}{2}\pi_{j_1} \ldots \mu^{22}_j \ldots \pi_{j+k} \ldots \pi_{j_p}$$
$$+ \tfrac{1}{2}\pi_{j_1} \ldots \pi_j \ldots \mu^{22}_{j+k} \ldots \pi_{j_p}. \tag{27}$$

Omitting as usual in the S-normalisation the numerical coefficients and taking into account that:

$$S(\pi_j \oplus \mu_j^{22})\chi_j^m = \pi_j\chi_j^m \tag{28}$$

$$\pi_{j_1} \ldots \mu_j^{22} \ldots \pi_{j+k} \ldots \pi_{j_p} \in H^{(p)}_{\langle j_1 \ldots j+k \ldots j_p\rangle}$$

$$\pi_{j_1} \ldots \pi_j \ldots \mu_{j+k}^{22} \ldots \pi_{j_p} \in H^{(p)}_{\langle j_1 \ldots j \ldots j+k\rangle} \tag{29}$$

we come to the conclusion:

$$S_{\frac{1}{2}}[H^{(1)}_{\langle j\rangle}H^{(p)}_{\langle j_1 \ldots j_p\rangle} \oplus H^{(p)}_{\langle j_1 \ldots j_p\rangle}H^{(1)}_{\langle j\rangle}]\Psi$$

$$\in \pi_{j_1} \ldots \pi_j \ldots \pi_{j+k} \ldots \pi_{j_p}[\theta_{j_1 \ldots j+k \ldots j_p,j}\theta_{j,j+k} + \theta_{j_1 \ldots j+k \ldots j_p,j}$$

$$+ \theta_{j,j+k} + H^{(p)}_{\langle j_1 \ldots j+k \ldots j_p\rangle} + H^{(p)}_{\langle j_1 \ldots j \ldots j_p\rangle}]\Psi. \tag{30}$$

Now it is clear that two last items in expression (30) can be omitted as their action has already been taken into account in the previous term

$$S_{\frac{1}{2}}[H^{(1)}_{\langle j\rangle}H^{(p-1)}_{\langle j_1 \ldots j_{p-1}\rangle} \oplus H^{(p-1)}_{\langle j_1 \ldots j_{p-1}\rangle}H^{(1)}_{\langle j\rangle}]\Psi$$

of series (23). Taking both that and property P3 into consideration and the fact that

$$\theta_{j_1 \ldots j+k \ldots j_p,j} + \theta_{j,j+k} = \Omega_{j_1 \ldots j \ldots j+k \ldots j_p} \tag{31}$$

we obtain

$$S_{\frac{1}{2}}[H^{(1)}_{\langle j\rangle}H^{(p)}_{\langle j_1 \ldots j_p\rangle} \oplus H^{(p)}_{\langle j_1 \ldots j_p\rangle}H^{(1)}_{\langle j\rangle}]\Psi = H^{(p+1)}_{\langle j_1 \ldots j \ldots j+k \ldots j_p\rangle}\Psi. \tag{32}$$

Summing up all conclusions concerning the items of expression (23) one can see that:

$$SH^{p+1}\Psi = S(H^p + H_{p+1})\Psi \tag{33}$$

which was to be proved.

Theorem. For every finite percolation lattice there is the such critical value of the order n_c that:

$$SH^{n_c+1}\Psi = SH^{n_c}\Psi. \tag{34}$$

It follows clearly from expression (23) that the operators H of powers $n+1$ and n differ from each other by the operator:

$$H_{n+1} = \sum_{\langle j_1 \ldots j_{n+1}\rangle} H^{(n+1)}_{\langle j_1 \ldots j_{n+1}\rangle} \tag{35}$$

which contains the sum over the clusters, where every cluster consists of $n+1$ neighbouring sites opened for percolation. Let us consider the finite percolation lattice with N sites and let NP sites ($P < 1$) be potentially opened for percolation under given conditions. It is possible to conclude that the theorem is proved at least for $n_c = NP$ as potentially percolating clusters with $n = NP + 1$ are absent in the percolation lattice. In reality NP is the upper limit for n_c. Practically the potentially percolating sites do not form the united cluster in every percolation lattice. They are distributed among several clusters which differ from each other by their form, size and which are not the neighbouring ones (i.e. these clusters are the groups of neighbouring percolating sites where none is the nearest neighbour for the potentially percolating sites from other groups). This is the reason why $n_c \leqslant \max(N_k)$, where N_k is the number of sites in the k's cluster. At least it can be seen from the construction of Hamiltonian (6) that the

percolation lattice evolution (irrigation of the oil reservoir) is possible only for those sites which have the states χ_j^2 as nearest neighbours (i.e. the sites are occupied by the water). So only those clusters take part in the percolation lattice evolution whose neighbours are in the states χ_j^2 at the initial moment. Thus one can conclude finally, that in the real percolation lattice evolution $n_c = \max(N_k')$ where N_k' is the number of sites in the k's cluster which is connected to the sites irrigated at the initial moment.

3. The computer numerical modelling of the dynamic percolation

The suggested model of the oil displacement by the water was mathematically realised on the rectangular plane lattice which was numbered by two sets of indexes i and j. The percolation lattice states at different time moments $t = n\tau$ were considered. The percolation lattice state functions at those moments were defined by the expression:

$$\Psi(n\tau) = \sum_{l=0}^{n} S_l(l\tau, 0)\Psi(0). \tag{36}$$

The initial state was, for example, chosen in the form:

$$\Psi(0) = \prod_{j=1}^{N_y} \chi_{1j}^1 \prod_{i=2}^{N_x} \prod_{j=1}^{N_y} \chi_{ij}^2. \tag{37}$$

This corresponds to the situation when the percolation lattice irrigation starts from its upper horizontal border. The vertical borders were considered as the impermeable ones. The process of percolation through the lattice was treated as the reaching of the lower border by the water. For the given value of sites potentially opened for the oil displacement, $N_p = PN_xN_y$ was defined where N_x and N_y were the numbers of sites in the columns and lines of the percolation lattice. These N_p sites were randomly distributed on the percolation lattice with the help of computer.

The previous study of such percolation lattice characteristics as a function of P_c has shown that when P is equal to some critical value P_c the second-kind phase transition is such that the water percolates through the percolation lattice independently of its size. When P is close to P_c all characteristics of the percolation lattice (oil reservoir) begin to depend crucially on the difference $|P - P_c|$. In particular the amount of the displaced oil at the first percolation moment can be present in the following way:

$$Q_1 = \begin{cases} 0 & \text{when } P < P_{c1} \\ A_1(P - P_{c1})^{-\alpha_1} & \text{when } P \geqslant P_{c1} \end{cases} \tag{38}$$

where A_1 is the coefficient of proportionality and α_1 is the critical index.

The dynamic percolation model introduced above allows us to describe the evolution of the percolation lattice states in time (in the process of oil displacement by water). All the oil reservoir characteristics acquire the dependence on time t. In particular the amount of displaced oil $Q(P)$ is now described by the function $Q(P, t)$. The previously known [3, 4] percolation transition (38) takes place in the dynamic theory at the time t_1 so that $Q(P, t_1) \equiv Q_1(P)$. The study of the evolution of the oil reservoir states allows us to reveal once more the manifestation of the same phase transition at the time moment $t_2 \geqslant t_1$ when the further oil displacement is stopped under the same

external conditions. The total amount of the displaced oil $Q_2(P) = Q(P, t_2)$ reveals also the analogous critical behaviour:

$$Q_2(P) = \begin{cases} 0 & \text{when } P < P_{c2} \\ A_2(P - P_{c2})^{-\alpha_2} & \text{when } P \geqslant P_{c2}. \end{cases} \qquad (39)$$

Now we can note that the theorem of the previous section shows the existence of the time moment t_2 when the oil displacement process is stopped. It is clear from the general point of view that the critical probabilities and the critical indexes are the same in the both time moments t_1 and t_2, while our computer calculations give the values

$$P_{c1} = P_{c2} = P_c = 0.5983 \qquad \alpha_1 = \alpha_2 = \alpha = 0.1386 \qquad (40)$$

which are the same within the accuracy of calculations. The critical behaviour of type (38) and (39) with the same values of p and α can be expected for the other intermediate time moments $t_1 \leqslant t \leqslant t_2$. The time moments t_1 and t_2 define the boundaries of the time interval during which the phase transition is taking place. The illustration of the square lattice evolution in the dynamical percolation model is given in figure 2. This percolation lattice corresponds to $N_x = N_y = 60$ and to one of the possible random distributions of potentially percolating sites, the amount of which is defined by $P = 0.605$. In figure 2 the first phase transition takes place when $t_1 = 87\tau$ and $Q_1(0.605) = 0.5067$. This phase transition lasts till $t_2 = 100\tau$ and $Q_2(0.605) = 0.5447$. The percolation in the same lattice with $P = 0.605$ but with changed initial conditions is illustrated in figure 3. Here at the initial moment $t = 0$ only the centre of percolation lattice (square 3×3) is considered as the irrigated region and the fact of percolation is understood as the reaching of the lattice boundary bonds by the water. For this particular realisation of the distribution of potentially percolating sites on the percolation lattice we obtain:

$$t_1 = 54\tau \qquad Q_1(0.605) = 0.1935$$

$$t_2 = 101\tau \qquad Q_2(0.605) = 0.3206. \qquad (41)$$

However, these figures do not have the statistical validity which is inherent to their averaging result over the large number of numerical experiments. The curves of the probability distributions $W(Q_1)$ and $W(Q_2)$ are shown in figure 4. These functions give the probability to obtain the values Q_1 and Q_2 in the random realisation of the position of potentially percolated sites on the percolation lattice. These data have been obtained with the help of 5000 experiments, every experiment corresponding to the same value of $P = 0.605$ for the case of percolation from the centre of the lattice and to the random realisation of the positions of potentially percolating sites on the percolation lattice. Just as expected, the numerical experiments have revealed the universality of the probability distribution curves when the lattice sizes change and $P \sim P_c$.

Generally speaking these numerical experiments would have been carried out using only such models as invasion percolation with trappings. However, it is the way of the pure mathematical empiricism that the theory of phase transition is not used. The real development of the theory demands a definite mathematical apparatus in order to obtain analytical results similar to those which have been carried out in section 2.

In subsequent work we will undertake the analytical and numerical investigations of the percolation dynamics, not only close to the critical region but also far from it.
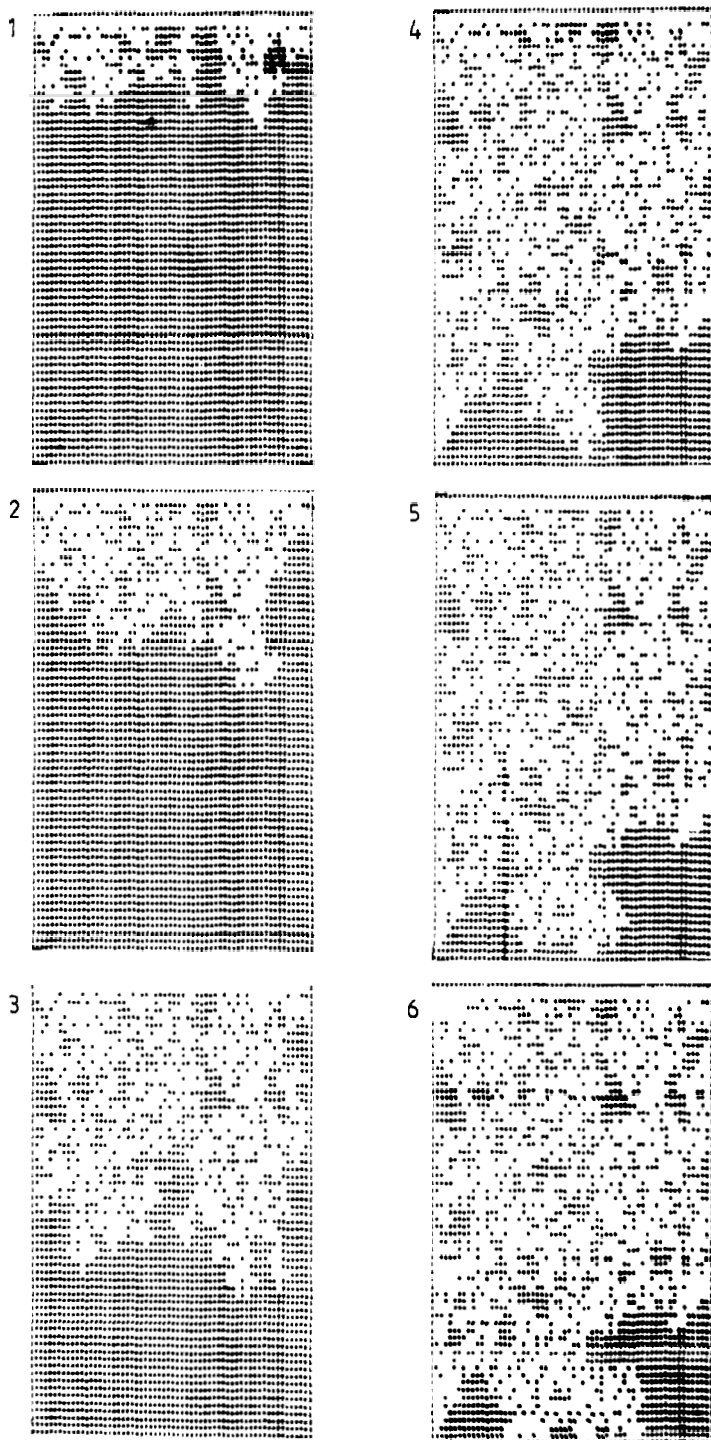
Figure 2. Evolution of the percolation lattice when the irrigation starts from the upper horizontal border ($N_x = N_y = 60$, $P = 0.605$).
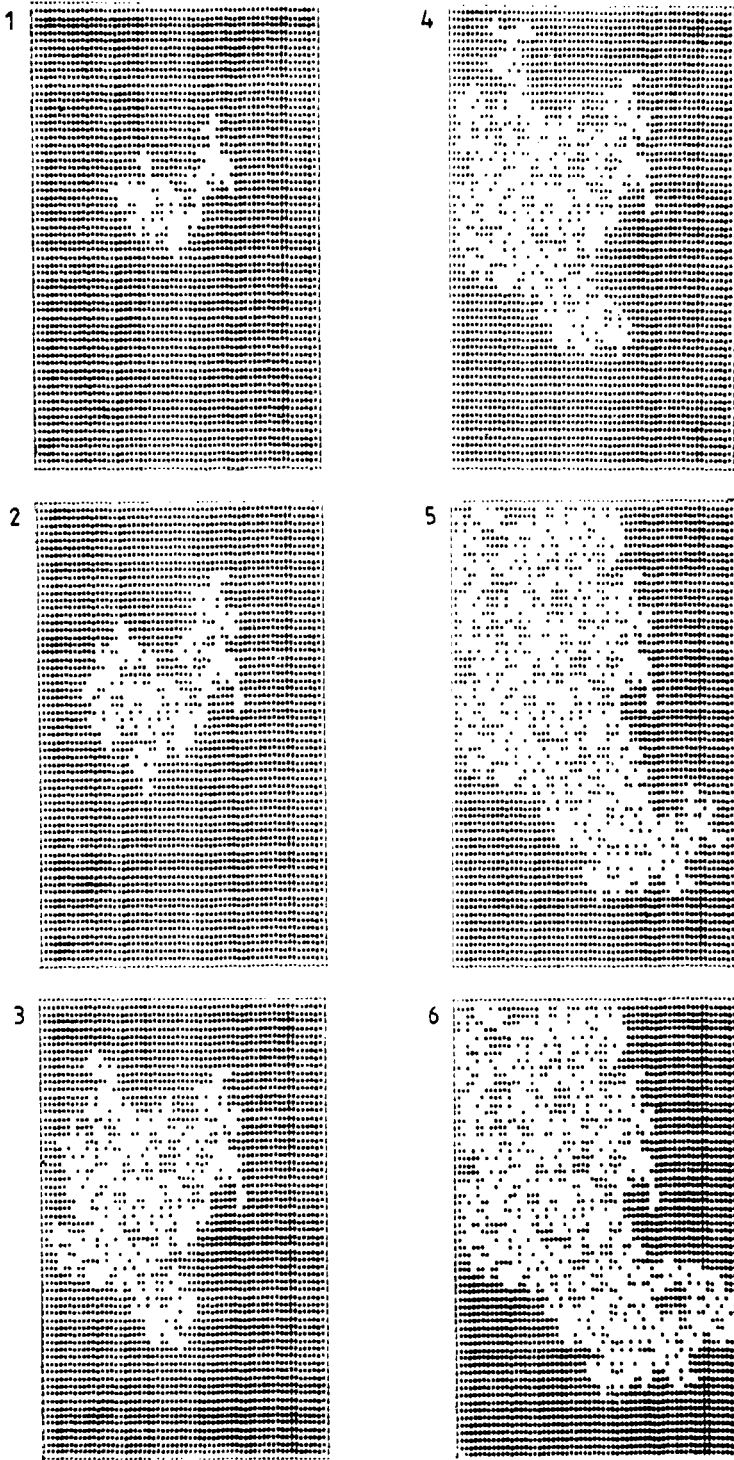
Figure 3. Evolution of the percolation lattice when the irrigation starts from the centre ($N_x = N_y = 60$, $P = 0.605$).
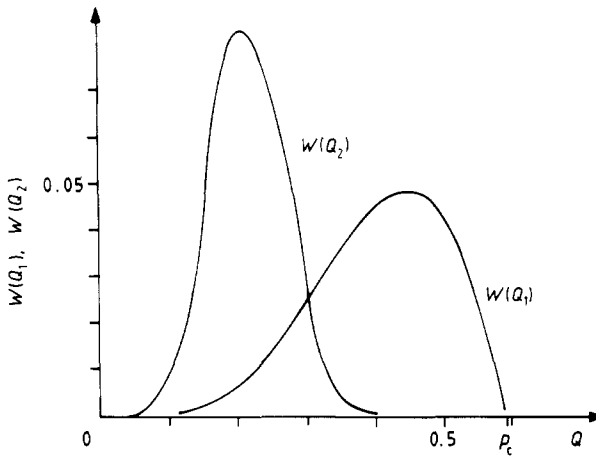
Figure 4. The distributions of displaced oil at time moments t_1 and t_2 when irrigation starts from the percolation lattice centre.

Appendix

We give here the commentaries and proofs of the properties P1–P6.

P1. This property results from the fact that the operators μ_j^{ab} matrix elements are positive integers in the basis χ_j^0 and from the method of construction of the operator H_j in the totality of the operators $\{\mu_j^{ab}\}$. The S-normalisation independence on the numerical state-items coefficients results from composition rules (15). For example,

$$S(\chi_j^1 \oplus 3\chi_j^2) = S(\chi_j^1 \oplus \chi_j^2) = \chi_j^2 \tag{A1}$$

and so on.

P2. This property can be easily proved by presenting the operator $t_p(\mu^{22})$ as the sum of kind (17) over the totality $\{j\}_L$. Then it turns out that one of the operators $t_p(\mu_{j_1}^{22}, \ldots, \mu_{j_p}^{22})$ enters into the square brackets with the coefficient 2 which can be omitted according to the property P1.

P3. It follows from the definitions D1 and D3 that the subspace $t_{p+q}(\mu^{22})\mathcal{H} \subset t_p(\mu^{22})\mathcal{H}$ so that the result of any operator function $f(\pi, \mu^{22})$ action on the subspace $t_{p+q}(\mu^{22})\mathcal{H}$ is contained in the subspace $f(\pi, \mu^{22})t_p(\mu^{22})\mathcal{H}$ and thus it can be omitted in P3 in the S-normalisation.

P4. According to definition D1 we have:

$$H_j^2 = \pi_j^2 \Omega_j^2 = \pi_j[t_1(\mu^{22}) + t_2(\mu^{22})] \tag{A2}$$

where $\{j_L\} = \langle j \rangle^*$. Now using P3 we obtain P4.

P5. For simplicity let us consider the proof of this property with the help of an example when $A = H_{\langle j_1 \rangle}^{(1)}$ and $B = H_{\langle j_2 \rangle}^{(1)}$ so that the operators $H_{\langle j_1 \rangle}^{(1)}$ and $H_{\langle j_2 \rangle}^{(1)}$ do not have the same sites in their definitions. Let us consider two auxiliary operators

$$Y_{j_1 j_2}^{k_1 k_2} = \pi_{j_1} \mu_{j_1 + k_1}^{22} \pi_{j_2} \mu_{j_2 + k_2}^{22} \qquad Z_{j_1 j_2}^{k_1 k_2} = \pi_{j_1} \mu_{j_1 + k_1}^{22} + \pi_{j_2} \mu_{j_2 + k_2}^{22} \tag{A3}$$

where the sites $j_1 + k_1$ and $j_2 + k_2$ are the nearest neighbours of the sites j_1 and j_2, respectively and

$$\pi_{j_1} \mu_{j_1+k_1}^{22} \in H_{\langle j_1 \rangle}^{(1)} \qquad \pi_{j_2} \mu_{j_2+k_2}^{22} \in H_{\langle j_2 \rangle}^{(1)}. \tag{A4}$$

The space of states for any pair of the percolation lattice sites, for example, for the sites $j_1 + k_1$ and $j_2 + k_2$, can be divided into four subspaces:

$$
\begin{aligned}
\Psi_{11} &= \chi_{j_1+k_1}^1 \chi_{j_2+k_2}^1 \Phi & \Psi_{12} &= \chi_{j_1+k_1}^1 \chi_{j_2+k_2}^2 \Phi \\
\Psi_{21} &= \chi_{j_1+k_1}^2 \chi_{j_2+k_2}^1 \Phi & \Psi_{22} &= \chi_{j_1+k_1}^2 \chi_{j_2+k_2}^2 \Phi.
\end{aligned}
\tag{A5}
$$

where Φ is any set of the percolation lattice rest sites. Let us act on the function Ψ_{22} by the operators $Y_{j_1 j_2}^{k_1 k_2}$ and $Z_{j_1 j_2}^{k_1 k_2}$:

$$Y_{j_1 j_2}^{k_1 k_2} \Psi_{22} = \chi_{j_1+k_1}^2 \chi_{j_2+k_2}^2 \pi_{j_1} \pi_{j_2} \Phi \tag{A6}$$

$$Z_{j_1 j_2}^{k_1 k_2} \Psi_{22} = \chi_{j_1+k_1}^2 \chi_{j_2+k_2}^2 (\pi_{j_1} + \pi_{j_2}) \Phi. \tag{A7}$$

Taking into account that the action of these operators π_j consists in the transformation of the functions χ_j^m with any m into the functions χ_j^2 and using composition rules (15) it is possible to obtain

$$SY_{j_1 j_2}^{k_1 k_2} \Psi_{22} = SZ_{j_1 j_2}^{k_1 k_2} \Psi_{22}. \tag{A8}$$

The action of these operators on the function Ψ_{11} is the same too:

$$Y_{j_1 j_2}^{k_1 k_2} \Psi_{11} = Z_{j_1 j_2}^{k_1 k_2} \Psi_{11} = 0. \tag{A9}$$

However, the action of the operators $Y_{j_1 j_2}^{k_1 k_2}$ and $Z_{j_1 j_2}^{k_1 k_2}$ on the functions Ψ_{12} and Ψ_{21} is not the same:

$$Y_{j_1 j_2}^{k_1 k_2} \Psi_{ab} = Z_{j_1 j_2}^{k_1 k_2} \Psi_{ab} \neq 0 \qquad (a, b = 1, 2). \tag{A10}$$

Thus it is possible to conclude that

$$SY_{j_1 j_2}^{k_1 k_2} \Psi \in SZ_{j_1 j_2}^{k_1 k_2} \Psi \tag{A11}$$

and since there are no restrictions in the choice of the indexes j_1, j_2, k_1 and k_2 we come to the conclusion that

$$SH_{\langle j_1 \rangle}^{(1)} H_{\langle j_2 \rangle}^{(1)} \Psi \in S[H_{\langle j_1 \rangle}^{(1)} + H_{\langle j_2 \rangle}^{(1)}] \Psi \in H\Psi \qquad (j_2 \neq j_1 + k_1). \tag{A12}$$

P6. For the proof of this property let us present the operator H^2 in the following form:

$$H^2 = \sum_{\langle j \rangle} H_j^2 + \sideset{}{'}\sum_{\langle j_1 \rangle \langle j_2 \rangle} H_{j_1} H_{j_2} + \sum_{\langle j_1 j_2 \rangle} \tfrac{1}{2}(H_{j_1} H_{j_2} + H_{j_2} H_{j_1}) \tag{A13}$$

where the prime on the second summation sign means that the sites j_1 and j_2 are not nearest neighbours.

Due to property P4 the first sum in expression (A13) is equivalent to the Hamiltonian

$$S\sum_{\langle j \rangle} H_j^2 \Psi = S\sum_{\langle j \rangle} H_j \Psi = SH\Psi. \tag{A14}$$

For the analysis of the second sum in expression (A13) let us use property P5 from which it follows that:

$$S \sideset{}{'}\sum_{\langle j_1 \rangle \langle j_2 \rangle} H_{j_1} H_{j_2} \Psi \in H\Psi \tag{A15}$$

where $H_j = H_{\langle j \rangle}^{(1)}$.

For the analysis of the third sum in expression (A13) it is convenient to present the operator Ω_j in the form:

$$\Omega_j = \theta_{j,j+k} + \mu_{j+k}^{22} \qquad \theta_{j,j+k} = \sum_{k' \neq k} \mu_{j+k'}^{22} \qquad (A16)$$

where the sum over k' is taken over the nearest neighbours of the site j with the exception of the site $j+k$. Now for the nearest sites j and $j+k$ we obtain

$$\tfrac{1}{2}(H_j H_{j+k} + H_{j+k} H_j)$$
$$= \pi_j \pi_{j+k} \theta_{j,j+k} \theta_{j+k,j} + \tfrac{1}{2} \pi_j \pi_{j+k} (\theta_{j,j+k} + \theta_{j+k,j})$$
$$+ \tfrac{1}{2} \pi_j \mu_{j+k}^{22} (1 + \theta_{j+k,j}) + \tfrac{1}{2} \pi_{j+k} \mu_j^{22} (1 + \theta_{j,j+k}). \qquad (A17)$$

Taking into account the properties of belonging:

$$\mu_j \mu_{j+k}^{22} \in H_j \qquad \pi_{j+k} \mu_j^{22} \in H_{j+k} \qquad \pi_j \pi_{j+k} \theta_{j,j+k} \theta_{j+k,j} \in H_j H_{j+k} \qquad (A18)$$

properties P1, P2 and expression (A12) it is possible to conclude that

$$S[\pi_j \pi_{j+k} \theta_{j,j+k} \theta_{j+k,j} + \tfrac{1}{2} \pi_j \mu_{j+k}^{22} (1 + \theta_{j+k,j}) + \tfrac{1}{2} \pi_{j+k} \mu_j^{22} (1 + \theta_{j,j+k})]\Psi = SH\Psi. \qquad (A19)$$

Now it is clear that the right-hand part of expression (A13) in the S-normalisation is equal to

$$S\left[H + \sum_{\langle j_1 j_2 \rangle} \pi_{j_1} \pi_{j_2} (\theta_{j_1,j_2} + \theta_{j_2,j_1}) \right] \Psi.$$

This fact, according to the definition

$$\theta_{j_1,j_2} + \theta_{j_2,j_1} = \Omega_{j_1 j_2} \qquad (A20)$$

proves property P6.

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